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Spherical-separability of Non-Hermitian Hamiltonians and Pseudo-$\mathcal{PT}$-symmetry

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Abstract Non-Hermitian but $\mathcal{P}_\varphi\mathcal{T}_\varphi$-symmetrized spherically-separable Dirac and Schrödinger Hamiltonians are considered. It is observed that the descendant Hamiltonians H_r, H_θ, and H_φ play essential roles and offer some "user-feriendly" options as to which one (or ones) of them is (or are) non-Hermitian. Considering a $\mathcal{P}_\varphi\mathcal{T}_\varphi$-symmetrized H_φ, we have shown that the conventional Dirac (relativistic) and Schrödinger (non-relativistic) energy eigenvalues are recoverable. We have also witnessed an unavoidable change in the azimuthal part of the general wavefunction. Moreover, setting a possible interaction $V(\theta) \neq 0$ in the descendant Hamiltonian H_θ would manifest a change in the angular θ-dependent part of the general solution too. Whilst some $\mathcal{P}_\varphi\mathcal{T}_\varphi$-symmetrized H_φ Hamiltonians are considered, a recipe to keep the regular magnetic quantum number m, as defined in the regular traditional Hermitian settings, is suggested. Hamiltonians possess properties similar to the $\mathcal{PT}$-symmetric ones (here the non-Hermitian $\mathcal{P}_\varphi\mathcal{T}_\varphi$-symmetric Hamiltonians) are nicknamed as *pseudo-$\mathcal{PT}$-symmetric*.

Keywords Non-Hermitian Hamiltonians · Spherical-separability · Pseudo-$\mathcal{PT}$-symmetry

1 Introduction

In the search for the reality conditions on the energy spectra/eigenvalues of non-Hermitian Hamiltonians [1–35], it is nowadays advocated (with no doubts) that the orthodoxal mathematical Hermiticity requirement to ensure the reality of the spectrum of a Hamiltonian is not only fragile but also physically deemed remote, obscure and strongly unnecessary. A tentative weakening of the Hermiticity condition through Bender's and Boettcher's [1] $\mathcal{PT}$-symmetric quantum mechanics (PTQM) (with $\mathcal{P}$ denoting parity and $\mathcal{T}$ time-reversal,

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a Hamiltonian H is $\mathcal{PT}$-symmetric if it satisfies $\mathcal{PT}H\mathcal{PT} = H$) has offered an alternative axiom that allows for the possibility of non-Hermitian Hamiltonians.

Such a PTQM theory, nevertheless, has inspired intensive research on the non-Hermitian Hamiltonians and led to the so-called pseudo-Hermitian Hamiltonians (i.e., a pseudo-Hermitian Hamiltonian H satisfies $\eta H \eta^{-1} = H^{\dagger}$ or $\eta H = H^{\dagger}\eta$, where η is a Hermitian invertible linear operator and $(^{\dagger})$ denotes the adjoint) by Mostafazadeh [16–21] which form a broader class of non-Hermitian Hamiltonians with real spectra and encloses within those $\mathcal{PT}$-symmetric ones. Moreover, not restricting η to be Hermitian (cf., e.g., Bagchi and Quesne [33]), and linear and/or invertible (cf., e.g., Solombrino [28], Fityo [29], and Mustafa and Mazharimousavi [30–32]) would weaken pseudo-Hermiticity and lead to real spectra.

Based on the inspiring example, nevertheless, by Bender, Brody and Jones [35] that $H = p^2 + x^2 + 2x$ is a non-$\mathcal{PT}$-symmetric whereas a simple amendment $H = p^2 + (x+1)^2 - 1$ (that leaves the Hamiltonian invariant and allows parity to perform reflection about $x = -1$ rather than $x = 0$) would consequently classify H as $\mathcal{PT}$-symmetric (i.e., reflection need not necessarily be through the origin) and promoting Znojil's understanding [34] of Bender's and Boettcher's PTQM (i.e., $\mathcal{P}$ and $\mathcal{T}$ need not necessarily mean just the parity and time reversal, respectively), we may introduce [36] a time-reversal-like,

$$\mathcal{T}_{\varphi} : \widehat{L}_z = -i\partial/\partial\varphi \longrightarrow -\widehat{L}_z = i\partial/\partial\varphi, \quad \varphi \longrightarrow \varphi,\ i \longrightarrow -i \tag{1}$$

and a parity-like

$$\mathcal{P}_{\varphi} : \widehat{L}_z \longrightarrow -\widehat{L}_z, \quad \varphi \longrightarrow (2\pi - \varphi), \tag{2}$$

operators that might very well be accommodated by Bender's and Boettcher's PTQM.

In this case, $\mathcal{P}_{\varphi}$ acting on a function $f(r,\theta,\varphi) \in L_2$ would read $\mathcal{P}_{\varphi} f(r,\theta,\varphi) = f(r,\theta,2\pi - \varphi)$. Moreover, $f(r,\theta,\varphi)$ is said to be $\mathcal{P}_{\varphi}\mathcal{T}_{\varphi}$-symmetric if it satisfies $\mathcal{P}_{\varphi}\mathcal{T}_{\varphi} f(r,\theta,\varphi) = f(r,\theta,\varphi)$. Hence, our new operators leave the coordinates r and θ unaffected and are designed to operate only on the azimuthal descendent eigenvalue equation (e.g., (21) below) of the spherically-separable non-Hermitian Hamiltonians (4) and (5). However, it should be noted that our parity-like operator $\mathcal{P}_{\varphi}$ in (2) is Hermitian, unitary, and performs reflection through a 2D-mirror represented by the xz-plane. Yet, the proof of the reality of the eigenvalues of a $\mathcal{P}_{\varphi}\mathcal{T}_{\varphi}$-symmetric Hamiltonian is straightforward. Let the eigenvalue equation of our $\mathcal{P}_{\varphi}\mathcal{T}_{\varphi}$-symmetric Hamiltonian be $H\psi(r,\theta,\varphi) = E\psi(r,\theta,\varphi)$, then $\mathcal{P}_{\varphi}\mathcal{T}_{\varphi}H\psi = \mathcal{P}_{\varphi}\mathcal{T}_{\varphi}E\psi = E\psi$. Using $[\mathcal{P}_{\varphi}\mathcal{T}_{\varphi}, H] = 0$ we obtain $E\psi = E^*\psi$ and E is therefore pure real (in analogy with Bender, Brody and Jones in [35] and fits into PTQM-recipe).

In the forthcoming proposal, using spherical coordinates, we depart from the traditional radial potential setting (i.e., $V(\mathbf{r}) = V(r)$) into a more general potential of the form

$$V(\mathbf{r}) = V(r,\theta,\varphi) = V(r) + \left[\frac{V(\theta) + V(\varphi)}{r^2 \sin^2\theta}\right]. \tag{3}$$

We shall use such potential setting in the context of Schrödinger Hamiltonian

$$H = -\nabla^2 + V(\mathbf{r}), \tag{4}$$

and within an equally-mixed vector, $V(\mathbf{r})$, and scalar, $S(\mathbf{r})$, potentials' setting in the Dirac Hamiltonian

$$H = \alpha \cdot \mathbf{p} + \beta[M + S(\mathbf{r})] + V(\mathbf{r}), \tag{5}$$

with the possibility of non-Hermitian interactions' settings in the process. However, it should be noted that such interactions in (3) with $V(r) = -\alpha/r$, $V(\theta) = -b^2$, and $V(\varphi) = 0$ represent just variants of the well known Hartmann potential [38–47] used in the studies of ring-shaped organic molecules.

For the sake of making our current proposal self-contained, we revisit, in Sect. 2, Dirac equation in spherical coordinates and give preliminary foundation on its separability. We connect, in the same section, Dirac descendant Hamiltonians with those of Schrödinger and provide a clear map for that. In Sect. 3, we explore some consequences of a class of complexified but $\mathcal{P}_\varphi\mathcal{T}_\varphi$-symmetrized azimuthal Hamiltonians. For a complexified azimuthal interaction $V(\varphi) \in \mathbb{C}$ (with $V(r), V(\theta) \in \mathbb{R}$) we use three illustrative examples for $V(\theta) = 0$, $V(\theta) = 1/2$, and $V(\theta) = 1/(2\cos^2\theta)$. In Sect. 4, a recipe of generating functions is provided to keep the magnetic quantum number as is, whenever deemed necessary of course. In the process of preserving the magnetic quantum number m, a set of isospectral φ-dependent potentials, $V(\varphi)$, for each set of $V(r)$ and $V(\theta)$ is obtained. This would, moreover, allow reproduction of the conventional-Hermitian relativistic and non-relativistic quantum mechanical eigenvalues within our $\mathcal{P}_\varphi\mathcal{T}_\varphi$-symmetric non-Hermitian settings. We give our concluding remarks in Sect. 5.

2 Separability and Preliminaries of Dirac and Schrödinger Equations Revisited

Dirac equation with scalar and vector potentials, $S(\mathbf{r})$ and $V(\mathbf{r})$, respectively, reads (in $\hbar = c = 1$ units)

$$\{\alpha \cdot \mathbf{p} + \beta[M + S(\mathbf{r})] + V(\mathbf{r})\}\psi(\mathbf{r}) = E\psi(\mathbf{r}), \tag{6}$$

where

$$\mathbf{p} = -i\nabla, \qquad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{7}$$

and $\boldsymbol{\sigma}$ is the vector Pauli spin matrix. A Pauli-Dirac representation would, with

$$\psi(\mathbf{r}) = \begin{pmatrix} \chi_1(\mathbf{r}) \\ \chi_2(\mathbf{r}) \end{pmatrix}, \tag{8}$$

yield the decoupled equations

$$(\boldsymbol{\sigma} \cdot \mathbf{p})\chi_2(\mathbf{r}) = [E - V(\mathbf{r}) - M - S(\mathbf{r})]\chi_1(\mathbf{r}), \tag{9}$$

$$(\boldsymbol{\sigma} \cdot \mathbf{p})\chi_1(\mathbf{r}) = [E - V(\mathbf{r}) + M + S(\mathbf{r})]\chi_2(\mathbf{r}). \tag{10}$$

An equally-mixed scalar and vector potentials (i.e., $S(\mathbf{r}) = V(\mathbf{r})$) leads to

$$\chi_2(\mathbf{r}) = \frac{\sigma \cdot \mathbf{p}}{E + M}, \tag{11}$$

and

$$[-\nabla^2 + 2(E + M)V(\mathbf{r})]\chi_1(\mathbf{r}) = [E^2 - M^2]\chi_1(\mathbf{r}). \tag{12}$$

Departing from the traditional "just-radially-symmetric" vector potential (i.e., $V(\mathbf{r}) = V(r)$) into a more general, though rather informative, vector potential (in the 3D spherical coordinates r, θ, and φ) of the form

$$V(\mathbf{r}) = V(r,\theta,\varphi) = V(r) + \left[\frac{V(\theta)+V(\varphi)}{r^2\sin^2\theta}\right], \tag{13}$$

would, with

$$\chi_1(\mathbf{r}) = \chi_1(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi), \tag{14}$$

imply

$$\begin{aligned}&\frac{1}{R(r)}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - 2(E+M)V(r)r^2 + (E^2-M^2)r^2\right\}R(r)\\&\quad+\frac{1}{\Theta(\theta)\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{2(E+M)V(\theta)}{\sin\theta}\right]\Theta(\theta)\\&\quad+\frac{1}{\Phi(\varphi)\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2} - 2(E+M)V(\varphi)\right)\Phi(\varphi) = 0\end{aligned} \tag{15}$$

The separability of which is obvious and mandates

$$\left\{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - \frac{\Lambda}{r^2} - 2(E+M)V(r) + (E^2-M^2)\right\}R(r) = 0, \tag{16}$$

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \left(\frac{2(E+M)V(\theta)+m^2}{\sin^2\theta}\right) + \Lambda\right]\Theta(\theta) = 0, \tag{17}$$

$$\left(\frac{d^2}{d\varphi^2} - 2(E+M)V(\varphi) + m^2\right)\Phi(\varphi) = 0, \tag{18}$$

where m^2 and Λ are separation constants to be determined below. Yet, in a straightforward manner, one can show that both Dirac and Schrödinger equations (with $V(\mathbf{r})$ in (13)) would read

$$\left\{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - \frac{\Lambda}{r^2} - V_{eff}(r) + \lambda\right\}R(r) = 0, \tag{19}$$

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \left(\frac{V_{eff}(\theta)+m^2}{\sin^2\theta}\right) + \Lambda\right]\Theta(\theta) = 0, \tag{20}$$

$$\left(\frac{d^2}{d\varphi^2} - V_{eff}(\varphi) + m^2\right)\Phi(\varphi) = 0, \tag{21}$$

where

$$V_{eff}(r) = \begin{cases} V(r) & \text{for Schrödinger,}\\ 2(E+M)V(r) & \text{for Dirac,}\end{cases} \tag{22}$$

$$V_{eff}(\theta) = \begin{cases} V(\theta) & \text{for Schrödinger,}\\ 2(E+M)V(\theta) & \text{for Dirac,}\end{cases} \tag{23}$$

$$V_{eff}(\varphi) = \begin{cases} V(\varphi) & \text{for Schrödinger,} \\ 2(E+M)V(\varphi) & \text{for Dirac,} \end{cases} \tag{24}$$

$$\lambda = \begin{cases} E & \text{for Schrödinger,} \\ E^2 - M^2 & \text{for Dirac.} \end{cases} \tag{25}$$

The map between Schrödinger and Dirac equations is clear, therefore. Moreover, one can safely name three "new" descendant Hamiltonians and recast the corresponding eigenvalue equations (with $\lambda = E$ for Schrödinger and $\lambda = E^2 - M^2$ for Dirac) as

$$H_r R(r) = \lambda R(r), \qquad H_\theta \Theta(\theta) = \Lambda \Theta(\theta), \qquad H_\varphi \Phi(\varphi) = m^2 \Phi(\varphi). \tag{26}$$

Of course it is a straightforward to work out the explicit forms of H_r, H_θ, and H_φ from (19), (20), and (21), respectively. Moreover, if we substitute $U(r) = R(r)/r$ in (19) then $U(0) = 0 = U(\infty)$. Yet, whilst $\Theta(0)$ and $\Theta(\pi)$ should be finite, $\Phi(\varphi)$ should satisfy the single-valuedness condition $\Phi(\varphi) = \Phi(\varphi + 2\pi)$. At this point, we argue that the reality of the spectrum of Dirac eigenvalue equation (6) is ensured not only by requiring $m, \Lambda, \lambda \in \mathbb{R}$ but also by requiring $\mathbb{R} \ni \lambda + M^2 = E^2 > 0$. With this understanding, we may now seek some $\mathcal{PT}$-symmetrization recipe (be it Lévai's [37] regular $\mathcal{PT}$-symmetrization or $\mathcal{P}_\varphi\mathcal{T}_\varphi$-symmetrization of Mazharimousavi [36]) for each (at a time) of the descendant Hamiltonians in (26).

3 Consequences of Complexified $\mathcal{P}_\varphi\mathcal{T}_\varphi$-symmetrized Azimuthal Hamiltonians

The eigenvalue equation in (21) with $a \in \mathbb{R}$ as a coupling parameter in a complexified-azimuthal effective interaction of the form

$$V_{eff}(\varphi) = -a^2 e^{i\varphi}, \tag{27}$$

would read

$$\left[\frac{d^2}{d\varphi^2} + a^2 e^{i\varphi} + m^2\right]\Phi(\varphi) = 0. \tag{28}$$

Hence, a change of variable of the form $z = e^{i\varphi/2}$ would result in

$$z^2 \frac{d^2\Phi(z)}{dz^2} + z\frac{d\Phi(z)}{dz} - (4m^2 + 4a^2 z^2)\Phi(z) = 0. \tag{29}$$

Obviously, (29) is the modified Bessel equation with imaginary argument and has two independent solutions. The linear combination of which reads the general solution

$$\Phi(z) = C_1 I_{2m}(2az) + C_2 K_{2m}(2az).$$

Each of these independent solutions should identically satisfy the single-valuedness condition $\Phi(\varphi) = \Phi(\varphi + 2\pi)$. One may, nevertheless, use the identities of [48] and closely follow Mazharimousavi's treatment (namely, (17)–(28) in [36]) and show that

$$I_{2m}(2ae^{i(\varphi+2\pi)/2}) = I_{2m}(2ae^{i\varphi/2}), \tag{30}$$

$$K_{2m}(2ae^{i(\varphi+2\pi)/2}) \neq K_{2m}(2ae^{i\varphi/2}). \tag{31}$$

Therefore, the regular solution collapses into

$$\Phi(z) = C_1 I_{2m}(2az) \quad \Longrightarrow \quad \Phi(\varphi) = C_1 I_{2m}(2ae^{i\varphi/2}). \tag{32}$$

Under such settings, it is obvious that the Hamiltonian represented in (28) reads

$$H_\varphi = -\frac{d^2}{d\varphi^2} - a^2 e^{i\varphi/2}, \tag{33}$$

and qualifies to be a $\mathcal{P}_\varphi \mathcal{T}_\varphi$-symmetric non-Hermitian Hamiltonian. That is,

$$\mathcal{P}_\varphi \mathcal{T}_\varphi H_\varphi \mathcal{P}_\varphi \mathcal{T}_\varphi = H_\varphi.$$

On the other hand, the eigenvalue equation (21) with H_φ would admit a regular azimuthal solution represented by the modified Bessel function

$$\Phi(\varphi) = C_{m,a} I_{2m}(2ae^{i\varphi/2}), \quad m = 0, \pm 1, \pm 2, \ldots \tag{34}$$

and satisfies the single-valuedness condition $\Phi(\varphi) = \Phi(\varphi + 2\pi)$ with $C_{m,a}$ as the normalization constant to be found through the relation

$$1 = \langle \Phi_m(\varphi) / \mathcal{P}_\varphi \mathcal{T}_\varphi \Phi_m(\varphi) \rangle = |C_{m,a}|^2 \int_0^{2\pi} |I_{2m}(2ae^{i\varphi/2})|^2 d\varphi. \tag{35}$$

Hereby, we have used the fact that our $\Phi_m(\varphi)$ in (34) is $\mathcal{P}_\varphi \mathcal{T}_\varphi$-symmetric satisfying $\mathcal{P}_\varphi \mathcal{T}_\varphi \Phi_m(\varphi) = \Phi_m(\varphi)$.

This would, in effect, suggest that since $\mathbb{R} \ni m = 0, \pm 1, \pm 2, \ldots$ and H_θ of (20) is therefore Hermitian, then H_θ of (20) admits real eigenvalues represented by $\Lambda \in \mathbb{R}$. Some illustrative consequences (with H_θ of (20) kept Hermitian) are in order.

3.1 Consequences of $V(\theta) = 0$ in (23)

Should $V(\theta) = 0$, one may clearly observe that (20) is the very well known associated Legendre equation in which $\Lambda = \ell(\ell + 1)$, where ℓ is the angular momentum quantum number, and $\Theta(\theta) = P_\ell^m(\cos\theta)$ are the associated Legendre functions. Hence, following the regular textbook procedure one may, in a straightforward manner, show that $\ell \geqslant |m|$ (i.e., $m = 0, \pm 1, \pm 2, \ldots, \pm \ell$, is the regular magnetic quantum number).

Consequently, as long as the Hermitian radial equation (19) is solvable (could be exactly-, quasi-exactly-, conditionally-exactly-solvable, etc.) for the radial interaction $V_{eff}(r)$, the spectrum remains invariant and real. However, the global wavefunction

$$\begin{aligned}\chi_1(r, \theta, \varphi) &= \psi_{Sch}(r, \theta, \varphi) \\ &= \sqrt{\frac{(2l+1)(l-|m|)!}{2(l+|m|)!}} C_{ma} R_{n_r l}(r) P_l^m(\cos\theta) I_{2m}(2ae^{i\varphi/2}),\end{aligned} \tag{36}$$

(with $n_r = 0, 1, 2, \ldots$ as the radial quantum number) would indulge some new probabilistic interpretations. This is due to the replacement of the regular spherical harmonics $Y_{\ell m}(\theta, \varphi)$ part (for the radially symmetric 3D-Hamitonians) by the new $\mathcal{P}_\varphi \mathcal{T}_\varphi$-symmetric part $P_\ell^m(\cos\theta)\Phi_m(\varphi)$ (defined above for our $\mathcal{P}_\varphi \mathcal{T}_\varphi$-symmetric non-Hermitian Hamiltonian model).

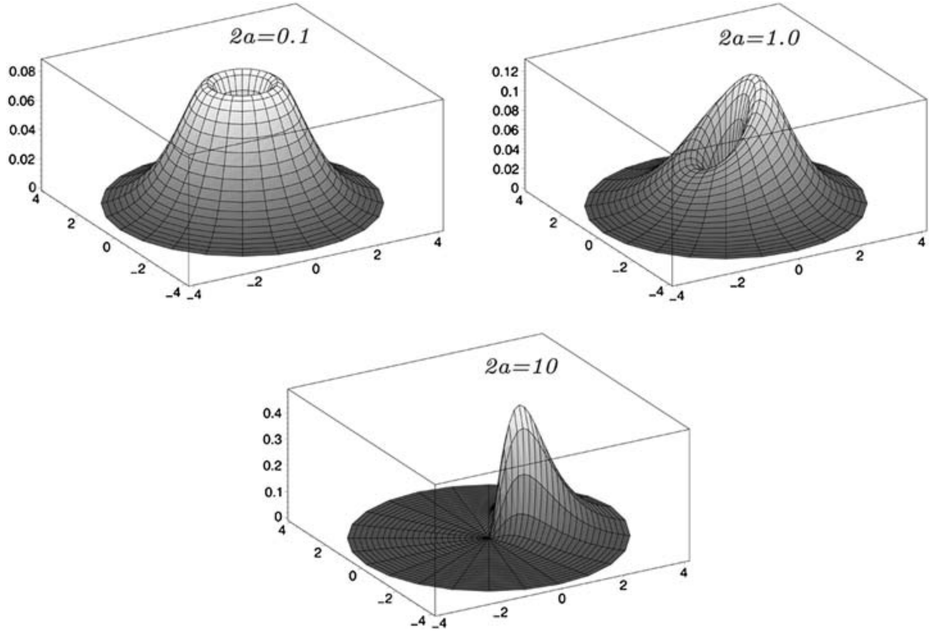

Fig. 1 Shows the effect of $V_{eff}(\varphi) = -a^2 e^{i\varphi}$ on the probability density as the coupling parameter a increases for $n = 1$, $\ell = 0 = m$

To see the effect of such a $\mathcal{P}_\varphi \mathcal{T}_\varphi$-symmetrization on the probability density, we consider a radial Coulombic effective interaction $V_{eff}(r) = -1/r$ accompanied by an azimuthal effective interaction $V_{eff}(\varphi) = -a^2 e^{i\varphi}$(an illustrative example of fundamental nature). In Figs. 1 and 2 we plot the corresponding probability densities at different values of the coupling parameter a for the principle quantum numbers $n = 1$ and $n = 2$ for $\ell = 0 = m$. It is clearly observed that whilst the probability density for small a imitates the Hermitian φ-independent probability density trends, it shifts and intensifies about $|\varphi| = 0$ as a increases (indicating that the corresponding state is more localized, therefore). In this case, of course, the rotational symmetry of a purely "just-radially-symmetric" Coulombic interaction breaks down as a result of $V_{eff}(\varphi)$.

3.2 Consequences of $V(\theta) = 1/2$ or $V(\theta) = 1/(2\cos^2\theta)$ in (23)

Taking $V(\theta) = 1/2$ in (23) would imply

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \frac{\widetilde{m}^2}{\sin^2\theta} + \Lambda\right]\Theta(\theta) = 0, \tag{37}$$

where $\widetilde{m} = \sqrt{E + M + m^2}$ for Dirac and $\widetilde{m} = \sqrt{1/2 + m^2}$ for Schrödinger settings. Similar equation was reported by Dutra and Hott [47]. The regular solution of which can (taking $\alpha = \beta = 0$ and $\gamma = 1$ in (12) of [47] to match with our settings) very well be copied and pasted to read (for Dirac equation)

$$\Theta(\theta) = y^\rho (1-y)^\upsilon {}_2F_1(-k, b; d; y), \qquad y = \cos^2(\theta/2), \tag{38}$$

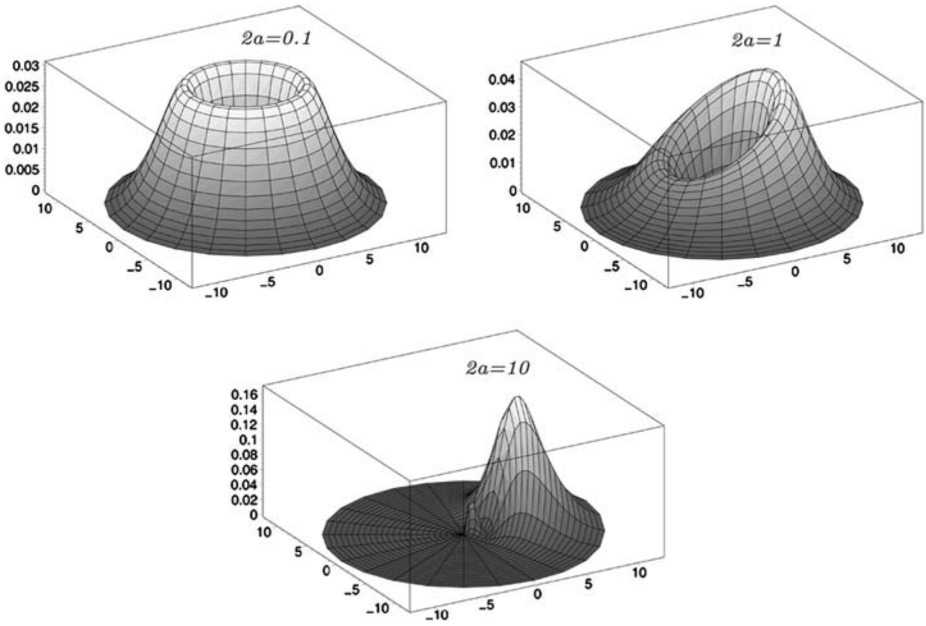

Fig. 2 Shows the effect of $V_{eff}(\varphi) = -a^2 e^{i\varphi}$ on the probability density as the coupling parameter a increases for $n = 2$, $\ell = 0 = m$

where

$$\upsilon = \rho = \frac{1}{2}\sqrt{m^2 + E + M}, \qquad b = k + 4\upsilon + 1, \qquad d = 1 + 2\upsilon, \tag{39}$$

$k = 0, 1, 2, \ldots$ is a "new" quantum number, and

$$\Lambda = \frac{1}{4}(b+k)^2 - \frac{1}{4} = \frac{1}{4}[(b+k+1)(b+k-1)]. \tag{40}$$

On the other hand, $V(\theta) = 1/(2\cos^2\theta)$ would (taking $\alpha = \beta = 0$ and $\gamma = 1$ in (13) of [12] for Dirac equation) result in

$$\rho = \frac{1}{4} + \frac{1}{4}\sqrt{1 + 4(E+M)}, \qquad \upsilon = \frac{1}{2}\sqrt{m^2 + E + M}, \tag{41}$$

$$b = k + 2(\rho + \upsilon) + \frac{1}{2}, \qquad d = 2\rho + \frac{1}{2} \tag{42}$$

and

$$\Lambda = (b+k)^2 - \frac{1}{4} = \left[\left(b + k + \frac{1}{2}\right)\left(b + k - \frac{1}{2}\right)\right]. \tag{43}$$

For Schrödinger case, nevertheless, one may just replace the term $(E + M)$ by $(1/2)$ in the above expressions and get the corresponding eigenvalue results. Then the general solution for both cases would read

$$\chi_1(r,\theta,\varphi) = \psi_{Sch}(r,\theta,\varphi) \\ = N_{n_r,k,m} R_{n_r,k}(r) y^{\rho}(1-y)^{\upsilon}{}_2F_1(-k,b;d;y) I_{2m}(2ae^{i\varphi/2}), \quad (44)$$

where $N_{n_r,k,m}$ is the normalization constant that can be obtained in a straightforward textbook procedure. Hereby, we witness that the general solution (44) exhibits the change not only in the azimuthal part but also in the angular θ-part.

4 Preservation of the Magnetic Quantum Number *m* and Isospectrality

To keep the magnetic quantum number as is (i.e., $m = 0, \pm 1, \pm 2, \ldots$), one may consider the azimuthal part of the general solution to be of the form

$$\Phi_m(\varphi) = e^{im\varphi} F(\varphi), \quad (45)$$

where $F(\varphi)$ satisfies the single-valuedness condition $F(\varphi) = F(\varphi + 2\pi)$.

Under such setting, the corresponding eigenvalue equation in (21) (with primes denoting derivatives with respect to φ) reads

$$F''(\varphi) + 2imF'(\varphi) - V_{eff}(\varphi)F(\varphi) = 0. \quad (46)$$

In this case $F(\varphi)$ would serve as a generating function for the sought after azimuthal potential $V_{eff}(\varphi)$ and shapes the form of the azimuthal solution $\Phi_m(\varphi)$. As an illustrative example, a generating function $F(\varphi) = \cos\varphi$ would imply

$$V_{eff}(\varphi) = -[1 + 2im\tan\varphi] \quad (47)$$

which is indeed a non-Hermitian and $\mathcal{P}_\varphi\mathcal{T}_\varphi$-symmetric, $\mathcal{P}_\varphi\mathcal{T}_\varphi V_{eff}(\varphi) = V_{eff}(\varphi)$.

However, one may wish to follow the other way around and consider a $\mathcal{P}_\varphi\mathcal{T}_\varphi$-symmetric $V_{eff}(\varphi)$ and solve (46) for $F(\varphi)$. In this manner, $V_{eff}(\varphi)$ would now serve as a generating function for $F(\varphi)$ and consequently a generating function for $\Phi_m(\varphi)$. An immediate example is in order. Consider

$$V_{eff}(\varphi) = -\frac{\omega^2}{4} e^{i\varphi} \quad (48)$$

and solve (44) for a regular $F(\varphi)$ to obtain

$$F(\varphi) = C_\circ e^{-im\varphi} I_{2m}(\omega e^{i\varphi/2}), \quad \mathbb{R} \ni m = 0, \pm 1, \pm 2, \ldots. \quad (49)$$

Then, (45) would read

$$\Phi_m(\varphi) = C_{m,\omega} I_{2m}(\omega e^{i\varphi/2}). \quad (50)$$

It is, therefore, obvious that all effective potentials $V_{eff}(\varphi)$ satisfying (48) would essentially change the azimuthal part of the general solution.

Moreover, in the process of preserving the magnetic quantum number m as defined in the regular Hermitian settings, a set of isospectral φ-dependent potentials, $V_{eff}(\varphi)$, is obtained. That is, for each set of $V_{eff}(r)$, $V_{eff}(\theta) \in \mathbb{R}$, all φ-dependent potentials, $V_{eff}(\varphi)$, satisfying (46) are isospectral.

5 Concluding Remarks

In the build up of a generalized quantum recipe (Bender's and Boettcher's PTQM in this case), a question of delicate nature arises in the process as to "would PTQM be able to recover some results (if not all, to be classified as a promising theory) of the conventional Hermitian quantum mechanics?". To the best of our knowledge, only rarely and mainly within regular Hermitian (but $\mathcal{PT}$-symmetric) settings examples were provided such as the one by Bender, Brody and Jones [35] mentioned in our introduction section above (i.e., $H = p^2 + x^2 + 2x$). The reality of the energy eigenvalues and other quantum mechanical properties (rather than the "recoverability of Hermitian quantum mechanical" results) were the main constituents and focal points in the studies of the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians. In our current proposal, with a new class of non-Hermitian $\mathcal{P}_\varphi\mathcal{T}_\varphi$-symmetric Hamiltonians (having real spectra identical to their Hermitian partner Hamiltonians), we tried to fill this gap, at least partially.

Through our over simplified non-Hermitian $\mathcal{P}_\varphi\mathcal{T}_\varphi$-symmetrized Hamiltonian (33), we have shown that some conventional relativistic and non-relativistic quantum mechanical results are indeed recoverable (the energy eigenvalues here). We have witnessed, however, an unavoidable change in the azimuthal part of the general wavefunction. Such a change would introduce some new probabilistic interpretations. With $V(\theta) = 0$ and $V_{eff}(\varphi) = -a^2 e^{i\varphi}$, for example, we have observed that a quantum state becomes more localized as the probability density intensifies at a specific point $|\varphi| = 0$ (documented in Figs. 1 and 2 of Sect. 3.1). Moreover, setting $V(\theta) \neq 0$ (again with $V_{eff}(\varphi) = -a^2 e^{i\varphi}$ in H_φ) in the descendant Hamiltonian H_θ has indeed manifested a change in the angular θ-dependent part of the general solution too (documented in Sect. 3.2). This would, of course, has some "new" effects on the probabilistic interpretations in turn. Yet, a recipe to keep the magnetic quantum number m as defined in the regular Hermitian quantum mechanical settings is suggested.

In connection with the current proposal's spherical-separability and non-Hermiticity, it is obvious that the descendant Hamiltonians H_r, H_θ, and H_φ play essential roles and offer some "user-friendly", say, options as to which one (or ones) of them is (or are) non-Hermitian. Be it $\mathcal{P}_\varphi\mathcal{T}_\varphi$-symmetric, $\mathcal{PT}$-symmetric, pseudo-Hermitian or η-pseudo-Hermitian, they very well fit into Bender's and Boettcher's PTQM (irrespective with their nicknames and with the understanding that $\mathcal{P}$ and $\mathcal{T}$ need not necessarily identify just parity and time reversal, respectively). Yet, a complexification of $0 \neq V(\theta) \in \mathbb{C}$ in H_θ with the understanding that a parity-like $\mathcal{P}_\theta$ and a time reversal-like $\mathcal{T}_\theta$ operators may very well suggest a similar $\mathcal{P}_\theta\mathcal{T}_\theta$-symmetric H_θ Hamiltonian. Such non-Hermitian $\mathcal{P}_\varphi\mathcal{T}_\varphi$-symmetrized and/or $\mathcal{P}_\theta\mathcal{T}_\theta$-symmetrized (anticipated to be feasible but yet to be identified) Hamiltonians better be nicknamed as *pseudo-$\mathcal{PT}$-symmetric Hamiltonians.*

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